Ostrogradsky's Hamilton formalism and quantum corrections

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Abstract

By means of a simple scalar field theory it is demonstrated that the Lagrange formalism and Ostrogradsky's Hamilton formalism in the presence of higher derivatives, in general, do not lead to the same results. While the two approaches are equivalent at the classical level, differences appear due to the quantum corrections.

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I. INTRODUCTION

The higher-derivative dynamics is of particular interest in the context of modern effective quantum field theories (see, e.g., Refs. [1, 2] and references therein). However, the quantization of Lagrangians involving higher derivatives is a non-trivial problem. The canonical quantization of field theories using the Hamilton formalism is a reliable method leading to a unitary scattering matrix. At the classical level, the Hamilton formalism for Lagrangians with higher derivatives was developed by Ostrogradsky [3] a long time ago. The canonical quantization based on Ostrogradsky's Hamilton formalism can be found, e.g., in Ref. [4].

In this work we examine Ostrogradsky's Hamilton formalism and demonstrate that this method, although equivalent to the Lagrange formalism at the classical level, may lead to wrong results due to the quantum corrections.

II. A TOY MODEL

We consider the following Lagrangian of two scalar fields A and Φ,

$$\mathcal{L}_1(A,\Phi) = \frac{1}{2}\partial_\mu A\partial^\mu A + \frac{1}{2}\partial_\mu \Phi\partial^\mu \Phi - \frac{M^2}{2}\,\Phi^2\,, \tag{1}$$

describing free massless (A) and massive (Φ) spinless particles. For simplicity, we do not include the dependence on the partial derivatives of the fields in the list of arguments of the Lagrangian. The momenta canonically conjugated to the fields A and Φ are defined by

$$p_A = \frac{\partial\mathcal{L}_1}{\partial\,\partial_0 A} = \partial_0 A\,, \tag{2}$$

$$p_\Phi = \frac{\partial\mathcal{L}_1}{\partial\,\partial_0 \Phi} = \partial_0 \Phi\,, \tag{3}$$

resulting in the Hamiltonian

$$p_A\partial_0 A + p_\Phi\partial_0\Phi - \mathcal{L}_1 = \frac{1}{2}p_A^2 + \frac{1}{2}\vec\nabla A\cdot\vec\nabla A + \frac{1}{2}p_\Phi^2 + \frac{1}{2}\vec\nabla \Phi\cdot\vec\nabla \Phi + \frac{1}{2}M^2\Phi^2 \equiv \mathcal{H}_1\,. \tag{4}$$

Let us consider the generating functional of the Green's functions of the field A in the canonical path integral representation,

$$Z[J] = \int \mathcal{D}A\mathcal{D}p_A\mathcal{D}\Phi\mathcal{D}p_\Phi\, e^{i\int d^4x\,(p_A\partial_0 A + p_\Phi\partial_0\Phi - \mathcal{H}_1 + J\,A)}\,, \tag{5}$$

where $\mathcal{H}_1$ is given in Eq. (4). In the following, we will repeatedly make use of a Gaussian functional integral of the form

$$\int \mathcal{D}p\, e^{i\int d^4x\left(-\frac{1}{2}p^2 + fp\right)} = \mathcal{N} e^{i\int d^4x\,\frac{1}{2}f^2}\,, \tag{6}$$

where $\mathcal{N}$ is an (irrelevant) multiplicative factor and f a given function not depending on p. Applying Eq. (6) to the p_A and p_Φ (functional) integrations in Eq. (5) and omitting, as is common practice, the corresponding multiplicative factors $\mathcal{N}$ of Eq. (6), yields

$$Z[J] = \int \mathcal{D}A\mathcal{D}\Phi\, e^{i\int d^4x\,[\mathcal{L}_1(A,\Phi) + J\,A]}\,, \tag{7}$$

where $\mathcal{L}_1$ is given in Eq. (1). The (full) propagator of the A field is given by

$$i\Delta_A(p) = \frac{i}{p^2 + i0^+}\,, \tag{8}$$

and the A field describes a massless non-interacting spinless particle.

III. FIELD TRANSFORMATIONS

Our line of arguments relies on the principle that, for a given theory, the physical content of the theory both at the classical as well as the quantum level should not depend on the choice of variables for describing the physical degrees of freedom. We will make use of the free theory described by the simple Lagrangian (Hamiltonian) of Eq. (1) [Eq. (4)] and its canonical path integral quantization described by Eq. (5). The path integral representation of Eq. (7) is a deduced quantity in the sense that it is derived from the canonical result of Eq. (5). The Green's functions obtained from Eq. (7), in particular the propagator of Eq. (8), will be taken as reference quantities. We will make use of two types of changes of field variables, namely, transformations without and with time derivatives of a field. Using the reference result of Sec. II, we will be able to point out that the canonical path integral quantization applied to a Hamiltonian based on the Ostrogradsky method, at the quantum level, does not describe an equivalent theory.

A. Field transformation without a time derivative

We first consider a change of field variables involving both spatial derivatives of a new field ϕ and the product of the A field with the ϕ field,

$$\Phi(x) = \phi(x) - c\,\Delta\phi(x) + g\,\phi(x)A(x)\,, \tag{9}$$

where (x) stands for $(t, \vec{x})$. In Eq. (9), the real parameters c and g carry the dimensions of a squared inverse mass and an inverse mass, respectively, and Δ denotes the usual Laplace operator. The Lagrangian density $\mathcal{L}_2$ in the new variables is obtained by substituting $\Phi(x)$ of Eq. (9) into the original Lagrangian $\mathcal{L}_1$ of Eq. (1),

$$\mathcal{L}_2(A,\phi) = \mathcal{L}_1(A,\Phi)\,. \tag{10}$$

Applying the change of variables, given by Eq. (9), directly to the generating functional of Eq. (7), we obtain

$$Z[J] \;=\; \int \mathcal{D}A\mathcal{D}\phi\,\det\left(\frac{\delta\Phi(y)}{\delta\phi(z)}\right) e^{i\int d^4x\,[\mathcal{L}_2(A,\phi)+J\,A]}\,, \tag{11}$$

where

$$\left(\frac{\delta\Phi(y)}{\delta\phi(z)}\right) = \Big([1 - c\,\Delta_y + g\,A(y)]\delta^4(y-z)\Big) \tag{12}$$

denotes the Jacobian "matrix" of the field transformation. Equation (11) is the generalization of the substitution rule for multiple Riemann integrals to functional integrals.

The same result as Eq. (11) for the generating functional is obtained by first applying the Hamilton formalism to the Lagrangian $\mathcal{L}_2$ of Eq. (10) and by subsequently performing the canonical path integral quantization. To that end we define the canonical momenta

$$p_A \;=\; \frac{\partial \mathcal{L}_2}{\partial\, \partial_0 A} = \partial_0 A + g\,\phi\,\partial_0\Phi\,, \tag{13}$$

$$p_\phi \;=\; \frac{\partial \mathcal{L}_2}{\partial\, \partial_0 \phi} = (1 - c\,\Delta + g\,A)\,\partial_0\Phi\,, \tag{14}$$

where

$$\partial_0\Phi = (1 - c\,\Delta + g\,A)\,\partial_0\phi + g\,\phi\,\partial_0 A\,. \tag{15}$$

From Eq. (14) we obtain

$$\partial_0\Phi = \widehat{O}p_\phi \equiv (1 - c\,\Delta + g\,A)^{-1}\,p_\phi\,. \tag{16}$$

Substituting Eq. (16) into Eq. (13), we can solve

$$\partial_0 A = p_A - g\,\phi\,\widehat{O}p_\phi\,. \tag{17}$$

Finally, inserting Eqs. (16) and (17) into Eq. (15), we can solve

$$\partial_0\phi = (1 - c\,\Delta + g\,A)^{-1}\left\{\left[1 + (g\,\phi)^2\right]\widehat{O}p_\phi - g\,\phi\,p_A\right\}\,. \tag{18}$$

Using integration by parts and omitting total divergences, the Hamiltonian takes the form

$$\begin{aligned}\mathcal{H}_2 \;&=\; p_A\partial_0 A + p_\phi\partial_0\phi - \mathcal{L}_2\\ &=\; \frac{1}{2}p_A^2 + \frac{1}{2}\left[1 + (g\,\phi)^2\right]\left(\widehat{O}p_\phi\right)^2 - g\,\phi\,\widehat{O}p_\phi\,p_A\\ &\quad+\frac{1}{2}\vec{\nabla}A\cdot\vec{\nabla}A + \frac{1}{2}\vec{\nabla}\Phi\cdot\vec{\nabla}\Phi + \frac{1}{2}M^2\Phi^2\,.\end{aligned} \tag{19}$$

In terms of the Hamiltonian $\mathcal{H}_2$ the generating functional for Green's functions of the A field is given by [4]

$$Z[J] = \int \mathcal{D}A\mathcal{D}p_A\mathcal{D}\phi\mathcal{D}p_\phi\, e^{i\int d^4x\,(p_A\partial_0 A + p_\phi\partial_0\phi - \mathcal{H}_2 + J\,A)}\,. \tag{20}$$

Introducing the new variable $\pi_\phi = \widehat{O}p_\phi$, we obtain

$$Z[J] = \int \mathcal{D}A\mathcal{D}p_A\mathcal{D}\phi\mathcal{D}\pi_\phi\,\det\left(\frac{\delta p_\phi(y)}{\delta\pi_\phi(z)}\right)e^{i\int d^4x\left(p_A\partial_0 A + \widehat{O}^{-1}\pi_\phi\partial_0\phi - \tilde{\mathcal{H}}_2 + J\,A\right)}\,, \tag{21}$$

where $\widehat{O}^{-1}\pi_\phi = (1 - c\,\Delta + g\,A)\,\pi_\phi$. The Jacobian matrix is given by

$$\left(\frac{\delta p_\phi(y)}{\delta\pi_\phi(z)}\right) = \Big([1 - c\,\Delta_y + g\,A(y)]\delta^4(y - z)\Big)\,,$$

and coincides with the Jacobian matrix of Eq. (12). Finally, the Hamiltonian $\tilde{\mathcal{H}}_2$ reads

$$\tilde{\mathcal{H}}_2 \;=\; \frac{1}{2}\,(p_A - g\,\phi\,\pi_\phi)^2 + \frac{1}{2}\pi_\phi^2 + \frac{1}{2}\vec{\nabla}A\cdot\vec{\nabla}A + \frac{1}{2}\vec{\nabla}\Phi\cdot\vec{\nabla}\Phi + \frac{1}{2}M^2\Phi^2\,. \tag{22}$$

By means of partial integration in the exponent of Eq. (21), the expression $\widehat{O}^{-1}\pi_\phi\partial_0\phi$ is replaced by $\pi_\phi\widehat{O}^{-1}\partial_0\phi$. Performing subsequently the p_A and π_ϕ integrations using Eq. (6), we obtain

$$Z[J] = \int \mathcal{D}A\mathcal{D}\phi \; \det\Big([1 - c\,\Delta_y + g\,A(y)]\,\delta^4(y-z)\Big)\, e^{i\int d^4x\,[\mathcal{L}_2(A,\phi)+J\,A]}\,, \tag{23}$$

which is identical with Eq. (11). To summarize this section, given the change of variables of Eq. (9) (without a time derivative), the substitution in the functional integral of Eq. (7) yields the same result as the application of the canonical path integral quantization starting from the Hamiltonian $\mathcal{H}_2$ derived from the Lagrangian $\mathcal{L}_2$.

B. Field transformation with time derivatives

We now go one step further and consider the following change of field variables involving time derivatives,

$$\Phi(x) = \chi(x) + c\,\Box\chi(x) + g\,\chi(x)\,A(x)\,, \tag{24}$$

where $\Box = \partial_0^2 - \Delta$ denotes the d'Alembert operator. The Lagrangian in the new variables is obtained from

$$\mathcal{L}_3(A,\chi) \;=\; \mathcal{L}_1(A,\Phi)\,. \tag{25}$$

Because of the d'Alembertian in the field transformation, the Lagrangian $\mathcal{L}_3(A,\chi)$ contains time derivatives of the field χ up to and including third order. Performing the change of variables in the generating functional of Eq. (7) results in

$$Z[J] = \int \mathcal{D}A\mathcal{D}\chi \; \det\left(\frac{\delta\Phi(y)}{\delta\chi(z)}\right) e^{i\int d^4x\,[\mathcal{L}_3(A,\chi)+J\,A]}\,, \tag{26}$$

with the Jacobian matrix

$$\left(\frac{\delta\Phi(y)}{\delta\chi(z)}\right) = \Big([1 + c\,\Box_y + g\,A(y)]\delta^4(y-z)\Big)\,.$$

We express the determinant of the Jacobian matrix in terms of a functional integral over ghost fields g_1 and g_2 (scalar Grassmann variables),

$$\det\Big([1 + c\,\Box_y + g\,A(y)]\delta^4(y-z)\Big) = \int \mathcal{D}g_1\mathcal{D}g_2\, e^{i\int d^4x\, g_2(1+c\Box+g\,A)g_1}\,. \tag{27}$$

In this representation the generating functional takes the following form,

$$Z[J] = \int \mathcal{D}A\mathcal{D}\chi\mathcal{D}g_1\mathcal{D}g_2\, e^{i\int d^4x\,[\mathcal{L}_3(A,\chi)+g_2\,(1+c\Box+g\,A)g_1+J\,A]}\,. \tag{28}$$

Using the example of the full propagator of the A field, we will illustrate that Eq. (28) gives rise to the same Green's functions including quantum corrections.

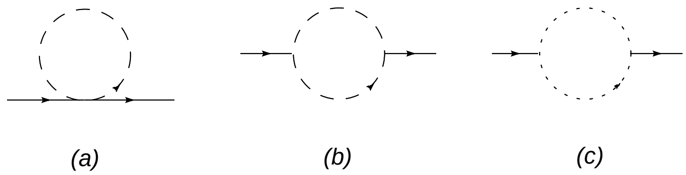

FIG. 1: One-loop contributions to the self energy of the A field. The solid, dashed, and dotted lines correspond to A, χ, and the ghost fields, respectively.

C. Full propagator of the A field at the one-loop level

We will investigate the propagator of the A field resulting from the perturbative expansion of Eq. (28) at the one-loop level. We will explicitly see that the quantum corrections obtained from Eq. (28) do not modify the position of the pole, i.e., the A field remains massless. The dressed propagator of the A field is of the form

$$i\Delta_A(p) = \frac{i}{p^2 - \Sigma(p^2)}\,, \tag{29}$$

where $-i\Sigma$ denotes the proper self-energy insertions of the A field, i.e., the sum of one-particle-irreducible diagrams contributing to the two-point function.

At the one-loop level, three diagrams contribute to the self energy (see Fig. 1). The corresponding Feynman rules are summarized in the Appendix. Using dimensional regularization, we obtain at $p^2 = 0$,

$$\Sigma^{(a)}(0) = -ig^2 \int \frac{d^n k}{(2\pi)^n} \frac{1}{(1 - c\,k^2)^2} = -\frac{1}{2}\Sigma^{(b)}(0) = \Sigma^{(c)}(0)\,. \tag{30}$$

These contributions cancel each other and, as a result, the quantum corrections do not give rise to a mass of the A field. This was expected, as the field transformation cannot change the physical content of a theory.

IV. OSTROGRADSKY'S HAMILTON FORMALISM

We now apply the Hamilton formalism and the canonical quantization to the Lagrangian of Eq. (25) and derive the generating functional. As the Lagrangian $\mathcal{L}_3$ contains time derivatives of higher orders, we use the Ostrogradsky formalism. While exactly the same results are obtained by following the procedure of Ref. [4], here we apply the method of Ref. [5] based on an auxiliary Lagrangian. We first define new independent fields

$$\begin{aligned} \psi &= \partial_0 \chi\,, \\ \zeta &= \partial_0 \psi\,. \end{aligned} \tag{31}$$

Introducing the Lagrange multipliers λ_1 and λ_2 in order to enforce the relations of Eq. (31), we obtain the auxiliary Lagrangian

$$\mathcal{L}_{\text{aux}} = \frac{1}{2}\,\partial_\mu A \partial^\mu A + \frac{1}{2}\,\left[\psi + c\,(\partial_0\zeta - \Delta\psi) + g\,\partial_0 A\,\chi + g\,A\,\psi\right]^2$$

$$
\begin{aligned}
&-\frac{1}{2}\,\vec{\nabla}\,[\chi + c\,(\zeta - \Delta\chi) + g\,A\,\chi]\cdot\vec{\nabla}\,[\chi + c\,(\zeta - \Delta\chi) + g\,A\,\chi] \\
&-\frac{1}{2}\,M^2\,[\chi + c\,(\zeta - \Delta\chi) + g\,A\,\chi]^2 + \lambda_1(\psi - \partial_0\chi) + \lambda_2(\zeta - \partial_0\psi)\,. \qquad (32)
\end{aligned}
$$

The momenta canonically conjugated to the degrees of freedom A, χ, ψ, ζ, λ_1, and λ_2 are defined as

$$
\begin{aligned}
p_A &= \frac{\partial\mathcal{L}_{\rm aux}}{\partial\,\partial_0 A} = \partial_0 A\left(1 + g^2\,\chi^2\right) + g\,\chi\,\left[\psi + c\,(\partial_0\zeta - \Delta\psi) + g\,A\,\psi\right]\,, \\
p_\chi &= \frac{\partial\mathcal{L}_{\rm aux}}{\partial\,\partial_0\chi} = -\lambda_1\,, \\
p_\psi &= \frac{\partial\mathcal{L}_{\rm aux}}{\partial\,\partial_0\psi} = -\lambda_2\,, \\
p_\zeta &= \frac{\partial\mathcal{L}_{\rm aux}}{\partial\,\partial_0\zeta} = c\,\left[\psi + c\,(\partial_0\zeta - \Delta\psi) + g\,\partial_0 A\,\chi + g\,A\,\psi\right]\,, \\
p_{\lambda_1} &= \frac{\partial\mathcal{L}_{\rm aux}}{\partial\,\partial_0\lambda_1} = 0\,, \\
p_{\lambda_2} &= \frac{\partial\mathcal{L}_{\rm aux}}{\partial\,\partial_0\lambda_2} = 0\,. \qquad (33)
\end{aligned}
$$

For $c \neq 0$, the two equations for p_A and p_ζ can be inverted to solve $\partial_0 A$ and $\partial_0\zeta$, respectively,

$$
\begin{aligned}
\partial_0 A &= p_A - \frac{1}{c}\,g\,\chi\,p_\zeta\,, \\
\partial_0\zeta &= \frac{1}{c^2}\left\{\left[1 + (g\,\chi)^2\right]p_\zeta - c\,g\,\chi\,p_A - c\,(1 - c\,\Delta + g\,A)\psi\right\}\,.
\end{aligned}
$$

The remaining velocities cannot be solved from Eqs. (33), i.e., the corresponding momenta need to satisfy the primary constraints

$$
\begin{aligned}
\Phi_1 &= p_{\lambda_1} \approx 0\,, \\
\Phi_2 &= p_{\lambda_2} \approx 0\,, \\
\Phi_3 &= p_\chi + \lambda_1 \approx 0\,, \\
\Phi_4 &= p_\psi + \lambda_2 \approx 0\,. \qquad (34)
\end{aligned}
$$

Here, $\Phi_i \approx 0$ denotes a weak equation in Dirac's sense, namely that one must not use one of these constraints before working out a Poisson bracket [6]. The so-called total or generalized Hamiltonian $\mathcal{H}^{(1)}$ has the form

$$
\mathcal{H}^{(1)} = \sum_{i=1}^{4}\Phi_i z_i + \mathcal{H}\,, \qquad (35)
$$

where

$$
\begin{aligned}
\mathcal{H} &= \frac{1}{2}\left[p_A - \frac{g\,\chi}{c}p_\zeta\right]^2 + \frac{1}{2}\,\frac{p_\zeta^2}{c^2} - \frac{1}{c}\,p_\zeta(1 - c\,\Delta + g\,A)\psi + \frac{1}{2}\,\vec{\nabla}A\cdot\vec{\nabla}A \\
&+\frac{1}{2}\,\vec{\nabla}\,[\chi + c\,(\zeta - \Delta\chi) + g\,A\,\chi]\cdot\vec{\nabla}\,[\chi + c\,(\zeta - \Delta\chi) + g\,A\,\chi] \\
&+\frac{1}{2}\,M^2\,[\chi + c\,(\zeta - \Delta\chi) + g\,A\,\chi]^2 - \lambda_1\psi - \lambda_2\zeta\,. \qquad (36)
\end{aligned}
$$

In Eq. (35), the z_i are arbitrary functions which have to be determined. The constraints of Eq. (34) have to be conserved in time. Therefore, we demand that the Poisson brackets of Φ_i with $H^{(1)} = \int d^3x\, \mathcal{H}^{(1)}$ vanish. An explicit evaluation of the Poisson brackets yields

$$
\begin{aligned}
\left\{\Phi_1, H^{(1)}\right\} \approx 0 \;\Rightarrow\; z_3 \;&=\; \psi\,, \\
\left\{\Phi_2, H^{(1)}\right\} \approx 0 \;\Rightarrow\; z_4 \;&=\; \zeta\,, \\
\left\{\Phi_3, H^{(1)}\right\} \approx 0 \;\Rightarrow\; z_1 \;&=\; -\frac{g}{c}\, p_\zeta \left(p_A - \frac{g\,\chi}{c}\, p_\zeta\right) \\
&\quad -(1 - c\,\Delta + g\,A)(\Delta - M^2)(\chi + c\,\zeta - c\,\Delta\,\chi + g\,\chi\,A)\,, \\
\left\{\Phi_4, H^{(1)}\right\} \approx 0 \;\Rightarrow\; z_2 \;&=\; -\frac{1}{c}\,(1 - c\,\Delta + g\,A)p_\zeta - \lambda_1\,.
\end{aligned}
$$

According to Ref. [4], the generating functional for the Green's functions of the A field can be written as a path integral over canonical coordinates and momenta,

$$
\begin{aligned}
Z[J] \;=\; &\int \mathcal{D}A\mathcal{D}p_A\mathcal{D}\chi\mathcal{D}p_\chi\mathcal{D}\lambda_1\mathcal{D}p_{\lambda_1}\mathcal{D}\lambda_2\mathcal{D}p_{\lambda_2}\mathcal{D}\psi\mathcal{D}p_\psi\mathcal{D}\zeta\mathcal{D}p_\zeta \\
&\times\delta\left[\Phi_1\right]\delta\left[\Phi_2\right]\delta\left[\Phi_3\right]\delta\left[\Phi_4\right]\left[\det\left(\{\Phi,\Phi\}\right)\right]^{\frac{1}{2}}\; e^{i\mathcal{S}[J]}\,,
\end{aligned}
\tag{37}
$$

where

$$
\mathcal{S}[J] = \int d^4x\,\left(p_A\partial_0 A + p_\chi\partial_0\chi + p_{\lambda_1}\partial_0\lambda_1 + p_{\lambda_2}\partial_0\lambda_2 + p_\psi\partial_0\psi + p_\zeta\partial_0\zeta - \mathcal{H} + J\,A\right)\,. \tag{38}
$$

The delta functionals in Eq. (37) take care of the constraints of Eq. (34). The entries of the 4×4 matrix $(\{\Phi,\Phi\})$ are given by the Poisson brackets $\{\Phi_i,\Phi_j\}$ and, in the present case, the determinant reduces to a constant which will be omitted in the following.

In Eq. (37), because of the delta functionals, the integrations over λ_i and p_{λ_i} are straightforward and give rise to the intermediate result

$$
\begin{aligned}
Z[J] \;=\; &\int \mathcal{D}A\mathcal{D}p_A\mathcal{D}\chi\mathcal{D}p_\chi\mathcal{D}\psi\mathcal{D}p_\psi\mathcal{D}\zeta\mathcal{D}p_\zeta \\
&\times e^{i\int d^4x\left(p_A\partial_0 A + p_\chi\partial_0\chi + p_\psi\partial_0\psi + p_\zeta\partial_0\zeta - \frac{1}{2}\left[p_A - \frac{g\,\chi}{c}p_\zeta\right]^2 - \ldots - \frac{1}{2}M^2[\chi + c\,(\zeta - \Delta\chi) + g\,A\,\chi]^2 - p_\chi\psi - p_\psi\zeta + JA\right)}\,.
\end{aligned}
\tag{39}
$$

The momenta p_χ and p_ψ appear linearly in the exponent of Eq. (39). Therefore, the integrations over p_χ and p_ψ give rise to the delta functionals $\delta[\partial_0\chi - \psi]$, and $\delta[\partial_0\psi - \zeta]$, respectively. The ζ integration then results in the replacement $(\zeta, \partial_0\zeta) \to (\partial_0\psi, \partial_0^2\psi)$, and the ψ integration in $(\psi, \partial_0\psi, \partial_0^2\psi) \to (\partial_0\chi, \partial_0^2\chi, \partial_0^3\chi)$. Finally, using Eq. (6) to perform the p_A and p_ζ integrations, and inserting Eq. (24), we obtain the following expression,

$$
\begin{aligned}
Z[J] \;&=\; \int \mathcal{D}A\mathcal{D}\chi\, e^{i\int d^4x\left[\frac{1}{2}(\partial_0 A)^2 - \frac{1}{2}\vec{\nabla}A\cdot\vec{\nabla}A + \frac{1}{2}(\partial_0\Phi)^2 - \frac{1}{2}\vec{\nabla}\Phi\cdot\vec{\nabla}\Phi - \frac{1}{2}M^2\Phi^2 + J\,A\right]} \\
&=\; \int \mathcal{D}A\mathcal{D}\chi\, e^{i\int d^4x\,[\mathcal{L}_3(A,\chi) + J\,A]}\,.
\end{aligned}
\tag{40}
$$

A comparison of Eq. (40) with Eq. (28) shows that the two results differ in the ghost part of the effective Lagrangian. Considering the dressed propagator of the A field generated by Eq. (40) we see that only the diagrams (a) and (b) in Fig. 1 contribute and hence their contributions at $p^2 = 0$ do not cancel. As a result the field A gains a non-vanishing mass due to the quantum corrections. This evidently contradicts the original physical content of the considered toy model.

V. CONCLUSIONS

In this work we discussed the quantization of field theories containing higher-order time derivatives of the fields in the Lagrangian. We started from a model describing one massless and one massive free spinless particle. The corresponding path-integral representation of the generating functional for the Green's functions of the massless field served as the reference point. We performed a change of field variables involving (time) derivatives resulting in a new Lagrangian containing higher-order time derivatives. To this Lagrangian we applied Ostrogradsky's Hamilton formalism for theories with higher derivatives and subsequently quantized the obtained theory using canonical quantization. The resulting generating functional for the Green's functions of the massless field differs from the one obtained by the change of variables in the reference generating functional. As a specific consequence, we showed that Ostrogradsky's formalism gives rise to a non-vanishing mass contribution for the massless particle due to the quantum corrections. These findings may be understood as follows. Ostrogradsky's formalism is equivalent to the introduction of new non-physical degrees of freedom. At the classical level, for the field variables of the original Lagrangian, Ostrogradsky's formalism leads to equations of motion which are equivalent to the ones of the Lagrange formalism. On the other hand, the non-physical degrees of freedom result in non-trivial contributions at the level of quantum corrections. We conclude that Ostrogradsky's Hamilton formalism may lead to wrong results and therefore, in general, cannot be considered as a satisfactory basis for the quantization of systems described with Lagrangians involving higher derivatives.

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VI. APPENDIX

Up to a total divergence, the Lagrangian of Eq. (25) can be written as

$$\begin{aligned}\mathcal{L}_3(A,\chi) \;=\;& \frac{1}{2}\partial_\mu A\partial^\mu A-\frac{1}{2}(\chi+c\,\Box\chi)(\Box+M^2)(\chi+c\,\Box\chi)\\ &-g\,A\,\chi(\Box+M^2)(\chi+c\,\Box\chi)+\frac{1}{2}\,g^2\left[A^2(\partial_\mu\chi\partial^\mu\chi-M^2\chi^2)-\chi^2A\,\Box A\right]\,. \quad (41)\end{aligned}$$

In combination with the ghost contribution, Eq. (41) results in the following Feynman rules:

1. Internal line of an A field with momentum k:

$$\frac{i}{k^2+i0^+}\,.$$

2. Internal line of a χ field with momentum k:

$$\frac{i}{(k^2-M^2+i0^+)(1-c\,k^2)^2}\,.$$

3. Internal line from ghost fields with momentum k:

$$\frac{i}{1 - c\,k^2}\,.$$

4. Vertex $\chi(p_i) \to \chi(p_f) + A$: $ig[(1 - c\,p_i^2)(p_i^2 - M^2) + (1 - c\,p_f^2)(p_f^2 - M^2)]$.

5. Vertex $\chi(p_i) + A(k_i) \to \chi(p_f) + A(k_f)$: $ig^2[2(p_i \cdot p_f - M^2) + k_i^2 + k_f^2]$.

6. Vertex $g_1 \to g_2 + A$: ig.

7. Because of the Grassmann nature of the ghost fields, a ghost loop produces an overall minus sign.

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